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Slow modes in passive scalar turbulent advection

Thomas Gilbert
Institut Non-Linéaire de Nice, CNRS, Université de Nice, 1361 Route des Lucioles, 06560 Valbonne, France
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This communication describes the computation of slow modes in a generic shell model of passive turbulent advection. It is argued that the propagator for the correlation functions possesses a ladder of slow modes associated with each zero mode. These slow modes decay algebraically fast, as opposed to exponential. Using the explicit form of the propagator of the 2-point structure function, we show that the slow modes structure for the generic case is analogous to the case of a Gaussian correlated advecting field, for which the differential structure of the operator allows a direct computation of these modes. Numerical computations of the slow modes are performed and compare very well to our results.

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Turbulent transport whereby a passive scalar or vector field is advected by a turbulent velocity field has recently been the subject of exciting new developments [1]. Indeed a fundamental consequence of the passive nature of such a transport process is that the evolution of the correlation functions of decaying fields may be expressed as an initial value problem under the form

$$\langle \phi(\vec{r}_1,t)\cdots\phi(\vec{r}_N,t)\rangle = \int d\mathbf{q}\mathcal{P}^{(N)}_{\mathbf{r},\mathbf{q}}(t|t_0)\langle\phi(\vec{q}_1,t_0)\cdots\phi(\vec{q}_N,t_0)\rangle, \tag{1}$$

where ϕ is here taken to be a scalar field and $\mathcal{P}^{(N)}_{\mathbf{r},\mathbf{q}}(t|t_0)$ is a linear operator propagating the Nth order correlation function from time t_0 at positions $\mathbf{q}\equiv(\vec{q}_1,\ldots,\vec{q}_N)$ to time t at positions $\mathbf{r}\equiv(\vec{r}_1,\ldots,\vec{r}_N)$. Here the brackets $\langle.\rangle$ indicate an average taken with respect to the statistics of both the initial conditions of the scalar and the realizations of the velocity field. The important point is that the right-hand side (RHS) of Eq. (1) expresses the decoupling between the initial distribution of the scalar and the statistics of the advecting field, a property that holds only at the initial time since correlations start building up between the advected and advecting fields at later times.

An important finding in the theory of turbulent transport was that the operator $\mathcal{P}^{(N)}$ has zero modes whose scaling properties determine the scaling of the correlation functions of the steady statistics of forced advection. That is, assuming stationary statistics for the advected field, its correlation functions have the property that

$$\langle\phi(\lambda\vec{r}_1)\cdots\phi(\lambda\vec{r}_N)\rangle_f = \lambda^{\zeta_N}\langle\phi(\vec{r}_1)\cdots\phi(\vec{r}_N)\rangle_f. \tag{2}$$

Here we dropped the time dependence and wrote $\langle.\rangle_f$ to denote an average with respect to the steady distribution of the forced advection problem. $\lambda>0$ and ζ_N are some numbers that cannot be determined by dimensional arguments; they are anomalous. It turns out the scaling exponents are a property of the zero modes of the decaying statistics.

The existence of the zero modes has been exemplified in [2] as follows. In the absence of a source term in the equations of motion of the passive field, its correlation functions are decaying objects. Nevertheless there exists special functions $Z^{(N)}(\mathbf{r})$ for which the quantities

$$I^{(N)}(t)\equiv\int d\mathbf{r}Z^{(N)}(\mathbf{r})\langle\phi(\vec{r}_1,t)\cdots\phi(\vec{r}_N,t)\rangle \tag{3}$$

remain constant in the appropriate limit; in other words, these functions are statistical integrals of motion. From Eq. (1) it follows that the functions $Z^{(N)}(\mathbf{r})$ are left eigenfunctions of $\mathcal{P}^{(N)}$ with unit eigenvalue [8]

$$Z^{(N)}(\mathbf{r}) = \int d\mathbf{q}\mathcal{P}^{(N)}_{\mathbf{q},\mathbf{r}}(t|t_0)Z^{(N)}(\mathbf{q}). \tag{4}$$

Such functions are the *zero modes* which stems from taking the time derivative of the RHS of Eq. (4),

$$\int d\mathbf{q}(d/dt)\mathcal{P}^{(N)}_{\mathbf{q},\mathbf{r}}(t|t_0)Z^{(N)}(\mathbf{q}) = 0;$$

in other words, a zero mode is an eigenmode of eigenvalue zero of the differential operator. These statements are only equivalent provided $\mathcal{P}^{(N)}$ can be written as the exponential of a differential operator, which is generally not the case.

The structure of the operator $\mathcal{P}^{(N)}$ is rather complicated and can be analyzed analytically only provided one makes drastic assumptions on the correlations of the advecting field. In particular, in the context of a Gaussian δ-correlated in time advecting field (so-called Kraichnan model [3]), it is possible to express $\mathcal{P}^{(N)}$ as the exponential of an operator, say $\mathcal{M}^{(N)}$, so that $\mathcal{P}^{(N)}$ has an explicit differential structure. As shown in [4], the zero modes of $\mathcal{P}^{(N)}$ are at the bottom of an infinite ladder of *slow modes*, which are simply eigenmodes of eigenvalue zero of the successive powers of $\mathcal{M}^{(N)}$. A zero mode of the jth power of $\mathcal{M}^{(N)}$ is a slow mode of $\mathcal{M}^{(N)}$ in the sense that the jth slow mode is projected onto the $(j-1)$th slow mode by the action of $\mathcal{M}^{(N)}$ and so on until it is projected on the zero mode $(j=1)$ at the $j-1$th iteration. The action of $\mathcal{P}^{(N)}$ on a slow mode thus yields an algebraic function of time whose degree is given by the position of the slow mode in the ladder. A consequence of this ladder structure is that the scaling exponents of the slow modes are obtained by simply taking multiples of the scaling exponent of the zero mode, i.e., if ζ_N is the scaling exponent of the zero mode $Z^{(N)}$, then $j\zeta_N$, $j=2,3,\ldots$ are the scaling exponents of the slow modes on that ladder.

In this paper, we intend to display the ladder structure of slow modes in a generic model of turbulent advection, extending the analysis of the zero modes done in [2]. That is to say we do not make any simplifying assumption on the statistics of the velocity field and compute the statistics of the passive field as generated by the advecting field with generic statistics.

The model we consider is a shell model defined on a space of scales associated with wave numbers $k_n = k_0\lambda^n$, indexed by $n \in \mathbb{Z}$. The advecting field consists of complex variables u_n associated to each shells whose evolution is specified by the sabra shell model [5],

$$\frac{d}{dt}u_n = i[k_{n+1}u_{n+1}^* u_{n+2} - \epsilon k_n u_{n-1}^* u_{n+1} + (1-\epsilon)k_{n-1}u_{n-2}u_{n-1}] - \nu k_n^2 u_n + f_n. \tag{5}$$

Here it will be assumed that $\epsilon = 1/2$ and $\lambda = 2$. The velocity forcing f_n is usually taken to be a Gaussian δ-correlated noise limited to two shells chosen on the large scales. This regime displays a direct cascade of energy, with nontrivial anomalous exponents for the structure functions of the velocity statistics.

Let θ_n be complex variables defining a scalar field advected by the u-field according to [6]

$$\frac{d}{dt}\theta_n = i(k_{n+1}\theta_{n+1}u_{n+1} + k_n\theta_{n-1}u_n^*) - \kappa k_n^2\theta_n + s_n. \tag{6}$$

Here s_n is a stochastic source term for the scalar. The decaying case corresponds to the absence of such term. Assuming so, we can write Eq. (6) in the simple form

$$\frac{d}{dt}\theta_n = \mathcal{L}_{n,m}\theta_m, \tag{7}$$

where the matrix $\mathcal{L}$ is specified by the advecting field. The evolution of the statistics of the θ-field can be expressed in a way similar to Eq. (1) with $\mathcal{P}^{(N)}$ now defined as

$$\mathcal{P}^{(N)}_{\underline{n},\underline{m}}(t|t_0) = \langle R_{n_1,m_1}(t|t_0)\cdots R_{n_N,m_N}(t|t_0)\rangle, \tag{8}$$

$$R_{n,m}(t|t_0) \equiv T^+\left\{\exp\left[\int_{t_0}^{t} ds\mathcal{L}(s)\right]\right\}_{n,m}, \tag{9}$$

with T^+ the time-ordering operator. Here the notation $\underline{n}$ stands for the N-tuple $(n_1,\ldots,n_N)$.

Consider the second order structure functions. We argue that the slow modes are given by the functions $S_n^{(j)} \sim 2^{-j\zeta_2 n}$, with $j=1$ corresponding to the zero modes, $S_n^{(1)} = Z_n^{(2)} \sim 2^{-\zeta_2 n}$, which is to say

$$K^{(j)}(t) \equiv \sum_n \langle|\theta_n(t)|^2\rangle S_n^{(j)} = \sum_{n,m}\mathcal{P}^{(2)}_{n,m}(t)\langle|\theta_m(0)|^2\rangle S_n^{(j)} \tag{10}$$

are polynomials in t of degree $j-1$. Numerical computations of these quantities are shown in Fig. 1.

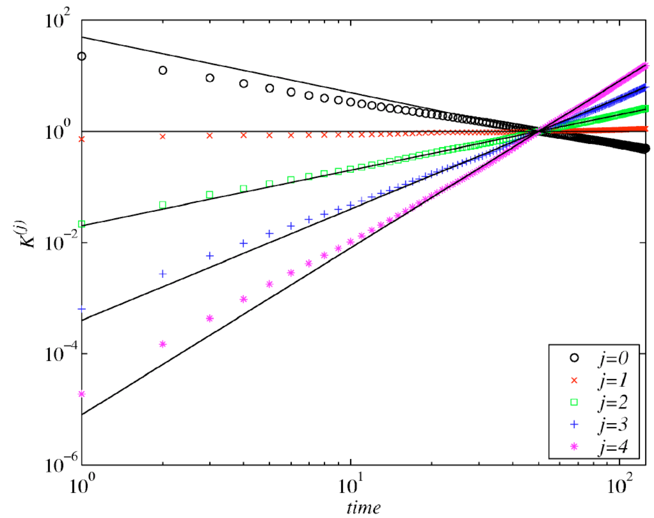

FIG. 1. (Color online) Numerical computation of Eq. (10) for the model described by Eqs. (5) and (6). The parameters are $\epsilon = 0.5$, $\lambda = 2$, $k_0 = 0.25$, $\nu = \kappa = 10^{-9}$. The number of shells used was 30, with Gaussian δ-correlated forcing on shells 2 and 3 for the velocity field. The initial condition for the scalar was initiated at shells 19 and 20 with random phases and constant amplitude. The average is taken over 13 000 realizations. The time units are natural, with a span corresponding to the eddy turn over the forcing scale (≈ 125). The first data set is simply $\Sigma_n\langle|\theta_n(t)|^2\rangle$. The other data sets correspond to the successive slow modes, starting from the zero mode, i.e., $\Sigma_n\langle|\theta_n(t)|^2\rangle F_n$, where F_n is the forced second order structure function, as obtained by direct numerical computation. The vertical scale is arbitrary and the curves have been scaled so as to cross at the same point.

In [7] the form of the propagator was shown to be (setting $t_0 = 0$)

$$P^{(2)}_{n,m}(t) \sim 2^{-(\zeta_2 m + \log_2 t)}H(\zeta_2 n + \log_2 t), \tag{11}$$

provided t is sufficiently large with respect to the local eddy turn over time of the initial scale. In this expression H is some homogeneous function of its argument.

Combining Eqs. (10) and (11), and assuming the initial condition $\langle|\theta_n(0)|^2\rangle = \delta_{n,n*}$, we can write

$$\begin{aligned} K^{(j)}(t) &= 2^{-\zeta_2 n^*}\sum_n 2^{-(j\zeta_2 n + \log_2 t)}H(\zeta_2 n + \log_2 t) \\ &= 2^{-\zeta_2 n^*}t^{j-1}\sum_n 2^{-j(\zeta_2 n + \log_2 t)}H(\zeta_2 n + \log_2 t) \sim t^{j-1}, \end{aligned} \tag{12}$$

which holds as long as we ignore boundary effects. This is indeed the power law displayed in Fig. 1.

These results can be compared to the derivation of slow modes in the context of a shell model where the passive scalar is advected by a Gaussian δ-correlated velocity field [6], $\langle u_n(t)u_m^*(t')\rangle = \delta_{n,m}\delta(t-t')2^{-\xi n}$. As shown in [7], in this context, the differential operator describing the propagation of the second order structure function is

$$\frac{d}{dt}\langle|\theta_n(t)|^2\rangle = \mathcal{M}^{(2)}_{n,m}\langle|\theta_m(t)|^2\rangle, \tag{13}$$

where

$$\mathcal{M}^{(2)}_{n,m} = 2[-(\tau^{-n} + \tau^{-(n+1)})\delta_{n,m} + \tau^{-n}\delta_{n,m-1} + \tau^{-(n+1)}\delta_{n,m+1}], \tag{14}$$

where $\tau = 2^{-(2-\xi)}$ [9]. Note that $\mathcal{M}^{(2)}$ is not a symmetric matrix. Therefore, we do not expect the left and right eigenvectors to be the same.

In fact, it is readily checked that τ^n is a left zero mode of $\mathcal{M}^{(2)}$, that is $\mathcal{M}^{(2)}_{n,m}\tau^n = 0$, while τ^{-n} is a right zero mode, $\mathcal{M}^{(2)}_{n,m}\tau^{-m} = 0$. More generally, one can check (j integer)

$$\mathcal{M}^{(2)}_{n,m}\tau^{jn} = 2\tau^{(j-1)m}(-1 - \tau^{-1} + \tau^{-j+1} + \tau^{j-2}), \tag{15}$$

$$\mathcal{M}^{(2)}_{n,m}\tau^{jm} = 2\tau^{(j-1)n}(-1 - \tau^{-1} + \tau^{j} + \tau^{-j-1}). \tag{16}$$

Therefore both τ^n and τ^{2n} are left zero modes of $\mathcal{M}^{(2)}$, while τ^{jn} ($j \geqslant 3$) are left slow modes in the sense that they are left zero modes of the $j-1$th power of $\mathcal{M}^{(2)}$. As far as multiplication on the right goes, we have two zero modes, namely the constant vector (this corresponds to a conservation law) and τ^{-n}. τ^{jn} ($j \geqslant 1$) are right slow modes, meaning they are right zero modes of the $j+1$th power of $\mathcal{M}^{(2)}$.

The exponentiation of $\mathcal{M}^{(2)}$ is the analog of the propagator $\mathcal{P}^{(2)}$ in Eq. (1),

$$\mathcal{P}^{(2)}_{n,m}(t) = \{\exp[t\mathcal{M}^{(2)}]\}_{n,m}. \tag{17}$$

Therefore, in this context, again assuming $\langle|\theta_n(0)|^2\rangle = \delta_{n,n^*}$, the quantity defined in Eq. (12) becomes

$$K^{(j)}(t) = \sum_n \mathcal{P}^{(2)}_{n,n*}(t)\tau^{jn} = \sum_{k=0}^{j-1}\frac{t^k}{k!}\sum_n [\mathcal{M}^{(2)^k}]_{n,n*}\tau^{jn}. \tag{18}$$

Here we used the fact that the exponential series stops since the operator is acting on a slow mode. Using Eq. (15), we compute

$$[\mathcal{M}^{(2)^k}]_{n,n*}\tau^{jn} = 2(-1 - \tau^{-1} + \tau^{-j+1} + \tau^{j-2})[\mathcal{M}^{(2)^{k-1}}]_{n,n*}\tau^{(j-1)n}$$
$$= 2^k\prod_{i=1}^{k}(-1 - \tau^{-1} + \tau^{-j+i} + \tau^{j-i-1})\tau^{(j-k)n*}. \tag{19}$$

The first few $K^{(j)}(t)$ are shown below and displayed in Fig. 2,

$$K^{(1)}(t) = \tau^{n*}, \tag{20}$$

$$K^{(2)}(t) = \tau^{2n*}, \tag{21}$$

$$K^{(3)}(t) = \tau^{3n*} + 2t\tau^{2n*}(-1 + \tau^{-2} - \tau^{-1} + \tau), \tag{22}$$

$$K^{(4)}(t) = \tau^{4n*} + 2t\tau^{3n*}(-1 + \tau^{-3} - \tau^{-1} + \tau^2) + 2t^2\tau^{2n*}(-1 + \tau^{-2} - \tau^{-1} + \tau)(-1 + \tau^{-3} - \tau^{-1} + \tau^2), \tag{23}$$

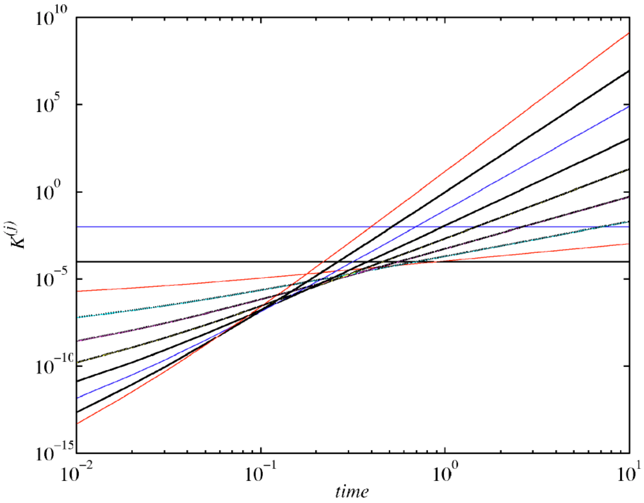

FIG. 2. (Color online) Numerical evaluation of the slow modes for the Kraichnan model on shells, Eqs. (20)–(25). Here $j = 1, \ldots, 10$. The parameter $\xi = 4/3$ and $n^* = 10$. The eddy turnover time at that shell is approximately 10^{-2}. The horizontal scale is counted in the units of the turnover time of shell 0.

$$K^{(5)}(t) = \tau^{5n*} + 2t\tau^{4n*}(-1 + \tau^{-4} - \tau^{-1} + \tau^3) + 2t^2\tau^{3n*}(-1 + \tau^{-3} - \tau^{-1} + \tau^2)(-1 + \tau^{-4} - \tau^{-1}\tau^3) + \tfrac{4}{3}t^3\tau^{2n*}(-1 + \tau^{-2} - \tau^{-1} + \tau)(-1 + \tau^{-3} - \tau^{-1} + \tau^2)(-1 + \tau^{-4} - \tau^{-1} + \tau^3), \tag{24}$$

$$K^{(6)}(t) = \tau^{6n*} + 2t\tau^{5n*}(-1 + \tau^{-5} - \tau^{-1} + \tau^4) + 2t^2\tau^{4n*}(-1 + \tau^{-4} - \tau^{-1} + \tau^3)(-1 + \tau^{-5} - \tau^{-1} + \tau^4) + \tfrac{4}{3}t^3\tau^{3n*}(-1 + \tau^{-3} - \tau^{-1} + \tau^2)(-1 + \tau^{-4} - \tau^{-1} + \tau^3)(-1 + \tau^{-5} - \tau^{-1} + \tau^4) + \tfrac{2}{3}t^4\tau^{2n*}(-1 + \tau^{-2} - \tau^{-1} + \tau)(-1 + \tau^{-3} - \tau^{-1} + \tau^2) \times(-1 + \tau^{-4} - \tau^{-1} + \tau^3)(-1 + \tau^{-5} - \tau^{-1} + \tau^4). \tag{25}$$

Obviously all the powers of t contribute to these expressions. However, if we assume t to be large enough with respect to the eddy turn over time of the initial condition, i.e., $t \gg \tau^{n*}$, then it can be seen that the time dependence of $K^{(j)}(t)$ is essentially dominated by the leading exponent, which is confirmed by Fig. 2 and is exactly what we see for the sabra advecting field in Fig. 1.

In conclusion, we have shown that shell models for passive advection offer nice grounds on which to extend analytical results derived in the framework of passive advection with very specific and seemingly restrictive assumptions made on the statistics of the velocity field. The same slow modes ladder structure that was originally derived by Bernard *et al.* [4] in the context of the Kraichnan model for the advecting field is seen to exist also for the shell models we considered, even though these models do not have a simple differential operator for the evolution of the structure functions. The observables introduced in this paper, which generalize the objects first defined in [2] for the sake of finding the zero modes, provide an easy way to single out a given slow mode among the whole ladder associated to a zero mode. The algebraic time dependence of the observables is indeed what one would expect should the propagator be expressible as the exponential of time multiplied by a differential operator. Despite the absence of an explicit form of the differential operator, the knowledge of the form of the propagator of the correlations functions derived in [7] is sufficient to infer the scaling properties of the slow modes

This work confirms that the ladder structure of zero and slow modes as described in [4] extends beyond the turbulent scalar advection by a velocity field given by the Kraichnan model.

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[8] Note that this does not mean that $\mathcal{P}^{(N)}$ admits an eigenvetor decomposition. For that sake one needs to properly compactify this operator.

[9] We assume all the quantities to be nondimensional and ignore all dissipative and finite size effects.